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Global weighted inequalities for operators and harmonic forms on manifolds

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Abstract

In this paper, both the local and global weighted Sobolev–Poincaré imbedding inequalities and Poincaré inequalities for the composition $T \circ G$ are established, where T is the homotopy operator and G is Green's operator applied to A-harmonic forms on manifolds.
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1. Introduction and notations

The investigation of the harmonic forms has developed rapidly during the recent years. The earlier studies of harmonic forms can be found in the work of Duff and Spencer [5,6]. Since then, the harmonic forms have been widely studied and used in many fields, including quasiconformal mappings, potential analysis and the theory of elasticity. Particularly, over the last several years, many interesting results of the harmonic forms and their applications in the fields such as partial differential equations, harmonic analysis and geometric function theory have been found, see [1–4,8,10,12,13,16]. Among these results, the Sobolev–Poincaré imbedding inequality and the Poincaré inequality are two critical ones which have engaged the attention of many mathematicians. On the other hand, Green's operator G and the homotopy operator T are two important operators which have become effective tools in different fields. In this paper, we first prove the local $A_r(M)$-weighted Sobolev–Poincaré imbedding inequalities and $A_r(M)$-weighted Poincaré inequalities for the composition of

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the homotopy operator and Green's operator acted on A-harmonic forms on manifolds in $\mathbf{R}^n$, $n \geqslant 2$. Then, we extend these inequalities into the global case. Our main results are presented in Theorems 2.8, 2.10, 3.3, 4.4 and 4.8, respectively. These inequalities can be used to study the integrability of A-harmonic forms and the properties of the related operators which are applied to the A-harmonic forms on manifolds.

We keep using the traditional notations in this paper. We always assume that M is a Riemannian, compact, oriented and C^∞ smooth manifold without boundary on $\mathbf{R}^n$ and Ω is an open subset of $\mathbf{R}^n$. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$ be the standard unit basis of $\mathbf{R}^n$ and $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ be the linear space of l-vectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$, $l = 0, 1, \ldots, n$. As usual, the Grassman algebra $\Lambda = \bigoplus_{l=0}^{n} \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \Lambda$ and $\beta = \sum \beta^I e_I \in \Lambda$, the inner product in Λ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$.

A differential l-form ω on M is a de Rham current (see [9,11]) on M with values in $\Lambda^l(\mathbf{R}^n)$. Let $\Lambda^l M$ be the lth exterior power of the cotangent bundle and $C^\infty(\Lambda^l M)$ be the space of smooth l-forms on M. We use $D'(M, \Lambda^l)$ to denote the space of all differential l-forms and $L^p(\Lambda^l M, w^\alpha)$ to denote the l-forms $\omega(x) = \sum_I \omega_I(x)\, dx_I = \sum \omega_{i_1 i_2 \ldots i_l}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ on M satisfying $\int_M |\omega_I|^p w^\alpha < \infty$ for all ordered l-tuples I, where w is a weight. In this way, $L^p(\Lambda^l M, w^\alpha)$ becomes a Banach space with norm

$$\|\omega\|_{p,M,w^\alpha} = \left(\int_M \left|\omega(x)\right|^p w^\alpha \, dx \right)^{1/p} = \left(\int_M \left(\sum_I \left|\omega_I(x)\right|^2 \right)^{p/2} w^\alpha \, dx \right)^{1/p},$$

where α is a real number. We write $L^p(\Lambda^l M) = L^p(\Lambda^l M, 1)$. The Hodge star operator $\star : \Lambda \to \Lambda$ is defined by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \Lambda$. The norm of $\alpha \in \Lambda$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbf{R}$. We denote the exterior derivative by $d : D'(M, \Lambda^l) \to D'(M, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n$. The Hodge codifferential operator $d^\star : D'(M, \Lambda^{l+1}) \to D'(M, \Lambda^l)$ is defined by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. We always use G to denote Green's operator and T to denote homotopy operator in this paper. Also, we use B to denote a ball and ρB, $\rho > 0$, is the ball with the same center as B and with $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. We do not distinguish the balls from cubes in this paper. For a measurable set $E \subset \mathbf{R}^n$, we write $|E|$ for the n-dimensional Lebesgue measure of E. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e.

In [8], Iwaniec and Lutoborski prove the following result: Let $D \subset \mathbf{R}^n$ be a bounded, convex domain. To each $y \in D$ there corresponds a linear operator $K_y : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})\, dt$$

and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$ holds at any $y \in D$. A homotopy operator $T : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ is defined by averaging K_y over all points y in D

$$T\omega = \int_D \varphi(y) K_y \omega \, dy, \tag{1.1}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)\,dy = 1$. We define the l-form $\omega_D \in D'(D, \wedge^l)$ by

$$\omega_D = |D|^{-1} \int_D \omega(y)\,dy, \quad l = 0, \quad \text{and} \quad \omega_D = d(T\omega), \quad l = 1, 2, \dots, n \tag{1.2}$$

for all $\omega \in L^p(D, \Lambda^l)$, $1 \leqslant p < \infty$, then $\omega_D = \omega - T(d\omega)$ and

$$\bigl\| d\bigl(T(u)\bigr)\bigr\|_{s,D} \leqslant \|u\|_{s,D} + C|D|\operatorname{diam}(D)\|du\|_{s,D}. \tag{1.3}$$

The A-harmonic equation for differential forms belongs to the nonlinear elliptic equations which take the form

$$d^\star A(x, d\omega) = 0, \tag{1.4}$$

where $A : M \times \Lambda^l(\mathbf{R}^n) \to \Lambda^l(\mathbf{R}^n)$ satisfies the following conditions:

$$\bigl|A(x,\xi)\bigr| \leqslant a|\xi|^{p-1} \quad \text{and} \quad \bigl\langle A(x,\xi), \xi\bigr\rangle \geqslant |\xi|^p$$

for almost every $x \in M$ and all $\xi \in \Lambda^l$ $(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(M, \Lambda^{l-1})$ such that $\int_M \langle A(x, d\omega), d\varphi\rangle = 0$ for all $\varphi \in W^{1,p}(M, \Lambda^{l-1})$ with compact support.

Definition 1.1. We call u an A-harmonic form or an A-harmonic tensor on a manifold M if u satisfies the A-harmonic equation on M.

2. The local Sobolev–Poincaré imbedding inequalities

We use the convention that $W^{1,p}(M, \Lambda^l)$ is the Sobolev space of l-forms which equals $L^p(\Lambda^l M) \cap L^p_1(\Lambda^l M)$ with norm

$$\|\omega\|_{W^{1,p}(M)} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M} + \|\nabla\omega\|_{p,M}. \tag{2.1}$$

Also, for $\omega \in D'(M, \Lambda^l)$, the vector-valued differential form $\nabla\omega = (\frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n})$ consists of differential forms $\frac{\partial\omega}{\partial x_i} \in D'(M, \Lambda^l)$, where the partial differentiation is applied to the coefficients of ω. The notations $W^{1,p}_{\mathrm{loc}}(M, \mathbf{R})$ and $W^{1,p}_{\mathrm{loc}}(M, \Lambda^l)$ are self-explanatory. The following weighted norm of $\omega \in W^{1,p}(M, \Lambda^l, w^\alpha)$ over M was introduced in [3]:

$$\|\omega\|_{W^{1,p}(M), w^\alpha} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M,w^\alpha} + \|\nabla\omega\|_{p,M,w^\alpha}, \tag{2.2}$$

where α is a real number, $0 < p < \infty$ and a weight $w(x)$.

We say that a differential form $u \in L^1_{\mathrm{loc}}(\Lambda^l M)$ has a generalized gradient if, for each coordinate system, the pullbacks of the coordinate function of u have generalized gradient in the familiar sense, see [14]. We write $\mathcal{W}(\Lambda^l M) = \{u \in L^1_{\mathrm{loc}}(\Lambda^l M)\colon u$ has generalized gradient$\}$. As usual, the harmonic l-fields are defined by $\mathcal{H}(\Lambda^l M) = \{u \in \mathcal{W}(\Lambda^l M)\colon du = d^\star u = 0, u \in L^p$ for some $1 < p < \infty\}$. The orthogonal complement of $\mathcal{H}$ in L^1 is defined by $\mathcal{H}^\perp = \{u \in L^1\colon \langle u, h\rangle = 0$ for all $h \in \mathcal{H}\}$. Green's operator G is defined as $G : C^\infty(\Lambda^l M) \to \mathcal{H}^\perp \cap C^\infty(\Lambda^l M)$ by assigning $G(u)$ to be the unique element of $\mathcal{H}^\perp \cap C^\infty(\Lambda^l M)$ satisfying Poisson's equation $\Delta G(\omega) = \omega - H(\omega)$, where H is either the harmonic projection or sometimes the harmonic part of ω. See [15] for more properties of Green's operator. From [13], we have the following lemma about L^s-estimates for Green's operator.

Lemma 2.1. *Let $u \in C^\infty(\Lambda^l M)$, $l = 0, 1, \ldots, n$. For $1 < s < \infty$, there exists a constant C, independent of u, such that*

$$\|dG(u)\|_{s,M} + \|d^\star G(u)\|_{s,M} + \|G(u)\|_{s,M} \leqslant C\|u\|_{s,M}. \tag{2.3}$$

We will use the following generalized Hölder inequality repeatedly in this paper.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,E} \leqslant \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E}$$

for any $E \subset \mathbf{R}^n$.

The following weak reverse Hölder inequality appears in [10].

Lemma 2.3. *Let u be an A-harmonic tensor in M, $\rho > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leqslant C|B|^{(t-s)/st}\|u\|_{t,\rho B}$$

for all balls or cubes B with $\rho B \subset M$.

Definition 2.4. We say that the weight $w(x)$ satisfies the $A_r(E)$ condition in a set $E \subset \mathbf{R}^n$ for $r > 1$, write $w \in A_r(E)$, if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty$$

for any ball $B \subset E$.

More properties of $A_r(E)$-weights and the proof of the following reverse Hölder inequality can be found in [7].

Lemma 2.5. *If $w \in A_r(E)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,B} \leqslant C|B|^{(1-\beta)/\beta}\|w\|_{1,B}$$

for all balls $B \subset E$.

The following results appear in [8].

Lemma 2.6. *Let $u \in L^s_{\mathrm{loc}}(B, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form in a ball $B \subset \mathbf{R}^n$. Then*

$$\|\nabla(Tu)\|_{s,B} \leqslant C|B|\|u\|_{s,B},$$
$$\|Tu\|_{s,B} \leqslant C|B|\operatorname{diam}(B)\|u\|_{s,B}.$$

First, we prove the following Sobolev–Poincaré imbedding inequality for the composition of the homotopy operator T and Green's operator G acted on a differential form.

Theorem 2.7. *Let $u \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be a smooth differential form on a manifold M and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be a homotopy operator defined in (1.1). If $w(x) \in A_r(M)$ for some $1 < r < \infty$, then there exists a constant C, independent of u, such that*

$$\big\|T\big(G(u)\big)\big\|_{W^{1,p}(B)} \leqslant C|B|\|u\|_{p,B} \tag{2.4}$$

for all balls B with $B \subset \mathbf{R}^n$.

Proof. Setting $\omega = T(G(u))$ in (2.1), we have

$$\big\|T\big(G(u)\big)\big\|_{W^{1,p}(B)} = \operatorname{diam}(B)^{-1}\big\|T\big(G(u)\big)\big\|_{p,B} + \big\|\nabla\big(T\big(G(u)\big)\big)\big\|_{p,B}. \tag{2.5}$$

Using Lemmas 2.1 and 2.6, we obtain

$$\big\|T\big(G(u)\big)\big\|_{p,B} \leqslant C_1|B|\operatorname{diam}(B)\big\|G(u)\big\|_{p,B} \leqslant C_2|B|\operatorname{diam}(B)\|u\|_{p,B} \tag{2.6}$$

and

$$\big\|\nabla\big(T\big(G(u)\big)\big)\big\|_{p,B} \leqslant C_3|B|\big\|G(u)\big\|_{p,B} \leqslant C_4|B|\|u\|_{p,B}. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we find that

$$\begin{aligned}\big\|T\big(G(u)\big)\big\|_{W^{1,p}(B)} &= \operatorname{diam}(B)^{-1}\big\|T\big(G(u)\big)\big\|_{p,B} + \big\|\nabla\big(T\big(G(u)\big)\big)\big\|_{p,B}\\ &\leqslant \operatorname{diam}(B)^{-1} \cdot C_2|B|\operatorname{diam}(B)\|u\|_{p,B} + C_4|B|\|u\|_{p,B}\\ &\leqslant C_5|B|\|u\|_{p,B}.\end{aligned}$$

This ends the proof of Theorem 2.7. □

Theorem 2.8. *Let $u \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be a smooth differential form satisfying (1.4) on a manifold M and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be a homotopy operator defined in (1.1). If $\rho > 1$ and $w(x) \in A_r(M)$ for some $1 < r < \infty$, then $T(G(u)) \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$. Moreover, there exists a constant C, independent of u, such that*

$$\big\|T\big(G(u)\big)\big\|_{p,B,w^\alpha} \leqslant C|B|\operatorname{diam}(B)\|u\|_{p,\rho B,w^\alpha} \tag{2.8}$$

for all balls B with $\rho B \subset M$ and any real number α with $0 < \alpha \leqslant 1$.

Proof. If (2.8) holds, then $T(G(u)) \in L^p_{\text{loc}}(\Lambda^l M, w^\alpha)$ follows. Thus we only need prove (2.8). We first show that (2.8) holds for $0 < \alpha < 1$. Let $t = p/(1-\alpha)$, then $p < t$. Using Lemma 2.2, we have

$$\begin{aligned}\big\|T\big(G(u)\big)\big\|_{p,B,w^\alpha} &= \bigg(\int_B \big(\big|T\big(G(u)\big)\big|w^{\alpha/p}\big)^p\,dx\bigg)^{1/p}\\ &\leqslant \bigg(\int_B \big|T\big(G(u)\big)\big|^t\,dx\bigg)^{1/t}\bigg(\int_B w^{t\alpha/(t-p)}dx\bigg)^{(t-p)/pt}\\ &= \big\|T\big(G(u)\big)\big\|_{t,B}\bigg(\int_B w\,dx\bigg)^{\alpha/p}. \end{aligned}\tag{2.9}$$

From (2.6), we see that

$$\big\|T\big(G(u)\big)\big\|_{t,B} \leqslant C_1|B|\operatorname{diam}(B)\|u\|_{t,B}. \tag{2.10}$$

Substituting (2.10) into (2.9) yields

$$\big\|T\big(G(u)\big)\big\|_{p,B,w^\alpha} \leqslant C_1|B|\operatorname{diam}(B)\|u\|_{t,B}\cdot\bigg(\int_B w\,dx\bigg)^{\alpha/p}. \tag{2.11}$$

Choose $s = p/(1+\alpha(r-1))$, then $s < p$. Applying Lemma 2.3, we find that

$$\begin{aligned}\big\|T\big(G(u)\big)\big\|_{p,B,w^\alpha} &\leqslant C_1|B|\operatorname{diam}(B)\|u\|_{t,B}\bigg(\int_B w\,dx\bigg)^{\alpha/p}\\ &\leqslant C_2|B|\operatorname{diam}(B)|B|^{(s-t)/st}\|u\|_{s,\rho B}\bigg(\int_B w\,dx\bigg)^{\alpha/p}. \end{aligned}\tag{2.12}$$

Using Lemma 2.2 with $1/s = 1/p + (p-s)/sp$, we have

$$\begin{aligned}\|u\|_{s,\rho B} &= \bigg(\int_{\rho B} |u|^s\,dx\bigg)^{1/s}\\ &= \bigg(\int_{\rho B} \big(|u|w^{\alpha/p}w^{-\alpha/p}\big)^s\,dx\bigg)^{1/s}\\ &\leqslant \bigg(\int_{\rho B} |u|^p w^\alpha\,dx\bigg)^{1/p}\bigg(\int_{\rho B}\bigg(\frac{1}{w}\bigg)^{\alpha s/(p-s)}dx\bigg)^{(p-s)/sp}\\ &= \|u\|_{p,\rho B,w^\alpha}\bigg(\int_{\rho B}\bigg(\frac{1}{w}\bigg)^{1/(r-1)}dx\bigg)^{\alpha(r-1)/p} \end{aligned}\tag{2.13}$$

for all balls B with $\rho B \subset M$. Substituting (2.13) into (2.12), we find that

$$\|T(G(u))\|_{p,B,w^\alpha} \leqslant C_2|B|\mathrm{diam}(B)|B|^{(s-t)/st}\|u\|_{p,\rho B,w^\alpha} \times \left(\int_B w\,dx\right)^{\alpha/p}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{\alpha(r-1)/p}. \tag{2.14}$$

Since $w \in A_r(M)$, we arrive at the estimate

$$\begin{aligned}
&\|w\|_{1,B}^{\alpha/p} \cdot \|1/w\|_{1/(r-1),\rho B}^{\alpha/p} \\
&\leqslant \left(\left(\int_{\rho B} w\,dx\right)\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right)^{\alpha/p} \\
&= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w\,dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right)^{\alpha/p} \\
&\leqslant C_3|B|^{\alpha r/p}.
\end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\begin{aligned}
\|T(G(u))\|_{p,B,w^\alpha} &\leqslant C_4|B|\mathrm{diam}(B)|B|^{(s-t)/st+\alpha r/p}\|u\|_{p,\rho B,w^\alpha} \\
&= C_5|B|\mathrm{diam}(B)\|u\|_{p,\rho B,w^\alpha}
\end{aligned}$$

for all balls B with $\rho B \subset M$. We have proved that (2.8) is true if $0 < \alpha < 1$. Next, we prove (2.8) is true for $\alpha = 1$. Let $t = p\beta/(\beta - 1)$, then $1 < p < t$ and $\beta = t/(t - p)$. Since $1/p = 1/t + (t - p)/pt$, by Lemma 2.2, we have that

$$\begin{aligned}
&\left(\int_B |T(G(u))|^p w\,dx\right)^{1/p} \\
&= \left(\int_B \left(|T(G(u))|w^{1/p}\right)^p dx\right)^{1/p} \\
&\leqslant \left(\int_B |T(G(u))|^t dx\right)^{1/t}\left(\int_B \left(w^{1/p}\right)^{pt/(t-p)} dx\right)^{(t-p)/pt} \\
&= \|T(G(u))\|_{t,B}\left(\int_B w^\beta\,dx\right)^{1/p\beta} \\
&= \|T(G(u))\|_{t,B}\|w\|_{\beta,B}^{1/p}.
\end{aligned} \tag{2.16}$$

By (2.6) and Lemma 2.5, we obtain

$$\begin{aligned}
\|T(G(u))\|_{p,B,w} &\leqslant \|T(G(u))\|_{t,B}\|w\|_{\beta,B}^{1/p} \\
&\leqslant C_6|B|\mathrm{diam}(B)\|u\|_{t,B}|B|^{(1-\beta)/\beta p}\|w\|_{1,B}^{1/p} \\
&= C_6|B|\mathrm{diam}(B)|B|^{-1/t}\|u\|_{t,B}\|w\|_{1,B}^{1/p}.
\end{aligned} \tag{2.17}$$

Choose $s = p/r$, then $s < p$. By Lemma 2.3, we have

$$\|u\|_{t,B} \leqslant C_7|B|^{(s-t)/st}\|u\|_{s,\rho B}. \tag{2.18}$$

Using Lemma 2.2 yields

$$\begin{aligned}\|u\|_{s,\rho B} &= \left(\int_{\rho B} \left(|u|w^{1/p}w^{-1/p}\right)^s dx\right)^{1/s} \\ &\leqslant \left(\int_{\rho B} |u|^p w\, dx\right)^{1/p} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{s/(p-s)} dx\right)^{(p-s)/sp} \\ &= \|u\|_{p,\rho B,w} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/p}.\end{aligned} \tag{2.19}$$

Notice that $w \in A_r(M)$, then

$$\begin{aligned}&\|w\|_{1,B}^{1/p} \cdot \|1/w\|_{1/(r-1),\rho B}^{1/p} \\ &\leqslant \left(\left(\int_{\rho B} w\, dx\right)\left(\int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/p} \\ &= \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B} w\, dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/p} \\ &\leqslant C_8|B|^{r/p}.\end{aligned} \tag{2.20}$$

Combining (2.17), (2.18), (2.19) and (2.20), we obtain

$$\begin{aligned}\left\|T\left(G(u)\right)\right\|_{p,B,w} &\leqslant C_9|B|\operatorname{diam}(B)|B|^{-1/t}\|w\|_{1,B}^{1/p}|B|^{(s-t)/st}\|u\|_{s,\rho B} \\ &\leqslant C_{10}|B|\operatorname{diam}(B)|B|^{-1/s}\|w\|_{1,B}^{1/p} \cdot \|1/w\|_{1/(r-1),\rho B}^{1/p}\|u\|_{p,\rho B,w} \\ &\leqslant C_{11}|B|\operatorname{diam}(B)|B|^{-1/s}\|u\|_{p,\rho B,w}|B|^{r/p} \\ &= C_{11}|B|\operatorname{diam}(B)\|u\|_{p,\rho B,w}\end{aligned}$$

for all balls B with $\rho B \subset M$. This ends the proof of Theorem 2.8. □

Similar to the proof of Theorem 2.8, we can prove the following theorem.

Theorem 2.9. *Let $u \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be a smooth differential form satisfying* (1.4) *on a manifold M and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be a homotopy operator defined in* (1.1). *If $\rho > 1$ and $w(x) \in A_r(M)$ for some $1 < r < \infty$, then $\nabla(T(G(u))) \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$. Moreover, there exists a constant C, independent of u, such that*

$$\left\|\nabla\left(T\left(G(u)\right)\right)\right\|_{p,B,w^\alpha} \leqslant C|B|\|u\|_{p,\rho B,w^\alpha} \tag{2.21}$$

for all balls B with $\rho B \subset M$ and any real number α with $0 < \alpha \leqslant 1$.

Now, we prove one of our main results, the local $A_r(M)$-weight Sobolev–Poincaré imbedding inequality for the composition of operators acted to A-harmonic forms on manifolds.

Theorem 2.10. *Let $u \in L^p_{\mathrm{loc}}(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be a smooth differential form satisfying (1.4) on a manifold M and $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ be a homotopy operator defined in (1.1). Assume that $\rho > 1$ and $w(x) \in A_r(M)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left\| T\big(G(u)\big)\right\|_{W^{1,p}(B),w^\alpha} \leqslant C|B|\|u\|_{p,\rho B,w^\alpha} \tag{2.22}$$

for all balls B with $\rho B \subset M$ and any real number α with $0 < \alpha \leqslant 1$.

Proof. Using (2.2), Theorems 2.8 and 2.9, we obtain

$$\begin{aligned}\left\| T\big(G(u)\big)\right\|_{W^{1,p}(B),w^\alpha} &= \operatorname{diam}(B)^{-1}\left\| T\big(G(u)\big)\right\|_{p,B,w^\alpha} + \left\|\nabla\big(T\big(G(u)\big)\big)\right\|_{p,B,w^\alpha} \\ &\leqslant \operatorname{diam}(B)^{-1}C_1\operatorname{diam}(B)|B|\|u\|_{p,\rho_1 B,w^\alpha} \\ &\quad + C_2|B|\|u\|_{p,\rho_2 B,w^\alpha} \\ &\leqslant C_3|B|\|u\|_{p,\rho B,w^\alpha}\end{aligned}$$

with $\rho = \max\{\rho_1, \rho_2\}$. We have completed the proof of Theorem 2.10. □

3. The local Poincaré inequalities

The purpose of this section is to establish the local Poincaré inequalities. From [2], we have Lemma 3.1 which will be used to prove Theorem 3.2.

Lemma 3.1. *Let $u \in C^\infty(\Lambda^l M)$, $l = 0, 1, \ldots, n$. If $1 < s < \infty$, then, there exists a constant C, independent of u, such that*

$$\left\| \big(G(u)\big)_D\right\|_{s,D} \leqslant C\|u\|_{s,D} \tag{3.1}$$

for any convex and bounded D with $D \subset M$.

Theorem 3.2. *Let $u \in L^s_{\mathrm{loc}}(B, \wedge^l)$, $l = 0, 1, \ldots, n$, be a differential form on a manifold M. Assume that $1 < s < \infty$, then there exists a constant C, independent of u, such that*

$$\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\right\|_{s,B} \leqslant C|B|\operatorname{diam}(B)\|u\|_{s,B}$$

for all balls B with $B \subset \mathbf{R}^n$.

Proof. If $1 \leqslant l \leqslant n$, from (1.2) and (1.3), we have

$$\begin{aligned}\left\| \big(T\big(G(u)\big)\big)_B\right\|_{s,B} &= \left\| d\big(T\big(T\big(G(u)\big)\big)\big)\right\|_{s,B} \\ &\leqslant \left\| T\big(G(u)\big)\right\|_{s,B} + C_1|B|\operatorname{diam}(B)\left\| d\big(T\big(G(u)\big)\big)\right\|_{s,B}.\end{aligned} \tag{3.2}$$

Then, by (1.2) and Lemma 3.1, we find that

$$\|d(T(G(u)))\|_{s,B} = \|(G(u))_B\|_{s,B} \leqslant C_2\|u\|_{s,B}. \tag{3.3}$$

Using Minkowski's inequality and combining (2.6), (3.2) and (3.3), we obtain

$$\begin{aligned}
&\|T(G(u)) - (T(G(u)))_B\|_{s,B} \\
&\quad \leqslant \|T(G(u))\|_{s,B} + \|(T(G(u)))_B\|_{s,B} \\
&\quad \leqslant 2\|T(G(u))\|_{s,B} + C_1|B|\operatorname{diam}(B)\|d(T(G(u)))\|_{s,B} \\
&\quad \leqslant C_3|B|\operatorname{diam}(B)\|u\|_{s,B} + C_1|B|\operatorname{diam}(B)\|d(T(G(u)))\|_{s,B} \\
&\quad \leqslant C_3|B|\operatorname{diam}(B)\|u\|_{s,B} + C_4|B|\operatorname{diam}(B)\|u\|_{s,B} \\
&\quad \leqslant C_5|B|\operatorname{diam}(B)\|u\|_{s,B}.
\end{aligned} \tag{3.4}$$

If $l = 0$, using (1.2), Lemma 2.2 with $1 = 1/s + 1/q$ and (2.6), we conclude that

$$\begin{aligned}
\|(T(G(u)))_B\|_{s,B} &= \left(\int_B (T(G(u)))_B^s \, dx\right)^{1/s} \\
&= \left(\int_B \left(\frac{1}{|B|}\int_B T(G(u(y)))\, dy\right)^s dx\right)^{1/s} \\
&\leqslant \left(\left(\frac{1}{|B|}\int_B |T(G(u(y)))|\, dy\right)^s \int_B 1\, dx\right)^{1/s} \\
&= \frac{1}{|B|}|B|^{1/s}\int_B |T(G(u(y)))|\, dy \\
&\leqslant |B|^{1/s-1}\left(\int_B |T(G(u(y)))|^s\, dy\right)^{1/s}\left(\int_B 1^q\, dy\right)^{1/q} \\
&= \|T(G(u))\|_{s,B} \leqslant C_6|B|\operatorname{diam}(B)\|u\|_{s,B}.
\end{aligned} \tag{3.5}$$

Applying Minkowski's inequality again and (3.5), we obtain

$$\begin{aligned}
\|T(G(u)) - (T(G(u)))_B\|_{s,B} &\leqslant \|T(G(u))\|_{s,B} + \|(T(G(u)))_B\|_{s,B} \\
&\leqslant C_7|B|\operatorname{diam}(B)\|u\|_{s,B}.
\end{aligned} \tag{3.6}$$

This ends the proof of Theorem 3.2. □

Now, we extend the above Theorem 3.2 into the local $A_r(M)$-weighted case.

Theorem 3.3. *Let $u \in L^s_{\mathrm{loc}}(B, \wedge^l)$, $l = 0, 1, \ldots, n$, be an A-harmonic form on a manifold M. Assume that $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u)) - (T(G(u)))_B\|_{s,B,w^\alpha} \leqslant C|B|\operatorname{diam}(B)\|u\|_{s,\rho B,w^\alpha} \tag{3.7}$$

for all balls B with $B \subset \mathbf{R}^n$ and any real number α with $0 < \alpha \leqslant 1$.

Proof. We first show that (3.7) holds for $0 < \alpha < 1$. Let $t = s/(1-\alpha)$, then $s < t$. Using Lemma 2.2 and Theorem 3.2, we obtain

$$\begin{aligned}
&\big\|T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\big\|_{s,B,w^\alpha} \\
&\quad = \bigg(\int_B \big(\big|T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\big| w^{\alpha/s}\big)^s\,dx\bigg)^{1/s} \\
&\quad \leqslant \bigg(\int_B \big|T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\big|^t\,dx\bigg)^{1/t} \bigg(\int_B w^{t\alpha/(t-s)}\,dx\bigg)^{(t-s)/st} \\
&\quad = \big\|T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\big\|_{t,B} \bigg(\int_B w\,dx\bigg)^{\alpha/s} \\
&\quad \leqslant C_1|B|\operatorname{diam}(B)\|u\|_{t,B}\|w\|_{1,B}^{\alpha/s}.
\end{aligned} \tag{3.8}$$

Let $m = s/(1+\alpha(r-1))$, then $m < s$. Applying Lemma 2.3 yields

$$\|u\|_{t,B} \leqslant C_2|B|^{(m-t)/mt}\|u\|_{m,\rho B}. \tag{3.9}$$

Using Lemma 2.2 with $1/m = 1/s + (s-m)/sm$, we have

$$\begin{aligned}
\|u\|_{m,\rho B} &= \bigg(\int_{\rho B} |u|^m\,dx\bigg)^{1/m} \\
&= \bigg(\int_{\rho B} \Big(|u| w^{\alpha/s} w^{-\alpha/s}\Big)^m\,dx\bigg)^{1/m} \\
&\leqslant \bigg(\int_{\rho B} |u|^s w^\alpha\,dx\bigg)^{1/s} \bigg(\int_{\rho B} \Big(\frac{1}{w}\Big)^{\alpha m/(s-m)}\,dx\bigg)^{(s-m)/sm} \\
&= \|u\|_{s,\rho B,w^\alpha} \bigg(\int_{\rho B} \Big(\frac{1}{w}\Big)^{1/(r-1)}\,dx\bigg)^{\alpha(r-1)/s} \\
&= \|u\|_{s,\rho B,w^\alpha}\|1/w\|_{1/(r-1),\rho B}^{\alpha/s}
\end{aligned} \tag{3.10}$$

for all balls B with $\rho B \subset M$. Combining (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned}
\big\|T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B\big\|_{s,B,w^\alpha} &\leqslant C_3|B|\operatorname{diam}(B)|B|^{(m-t)/mt}\|u\|_{s,\rho B,w^\alpha} \\
&\quad \times \|w\|_{1,B}^{\alpha/s} \cdot \|1/w\|_{1/(r-1),\rho B}^{\alpha/s}.
\end{aligned} \tag{3.11}$$

Since $w \in A_r(M)$, thus

$$
\begin{aligned}
&\|w\|_{1,B}^{\alpha/s} \cdot \|1/w\|_{1/(r-1),\rho B}^{\alpha/s} \\
&\quad \leqslant \left(\left(\int_{\rho B} w\,dx \right) \left(\int_{\rho B} \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{\alpha/s} \\
&\quad = \left(|\rho B|^r \left(\frac{1}{|\rho B|} \int_{\rho B} w\,dx \right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{\alpha/s} \\
&\quad \leqslant C_4 |B|^{\alpha r/s}.
\end{aligned} \tag{3.12}
$$

Substituting (3.12) into (3.11), we obtain that

$$
\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{s,B,w^\alpha} \leqslant C_5 |B| \operatorname{diam}(B) \|u\|_{s,\rho B,w^\alpha} \tag{3.13}
$$

for all balls B with $\rho B \subset M$. We have proved that (3.7) is true if $0 < \alpha < 1$. Now, we prove (3.7) is true for $\alpha = 1$, that is, we need to show that

$$
\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{s,B,w} \leqslant C |B| \operatorname{diam}(B) \|u\|_{s,\rho B,w}. \tag{3.14}
$$

By Lemma 2.5, there exist constants $\beta > 1$ and $C_6 > 0$, such that

$$
\|w\|_{\beta,B} \leqslant C_6 |B|^{(1-\beta)/\beta} \|w\|_{1,B} \tag{3.15}
$$

for all balls $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta - 1)$, then $1 < s < t$ and $\beta = t/(t-s)$. Using Lemma 2.2, Theorem 3.2 and (3.15), we have

$$
\begin{aligned}
&\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{s,B,w} \\
&\quad = \left(\int_B \left(\left| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right| w^{1/s} \right)^s dx \right)^{1/s} \\
&\quad \leqslant \left(\int_B \left| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right|^t dx \right)^{1/t} \left(\int_B w^{t/(t-s)}\,dx \right)^{(t-s)/st} \\
&\quad \leqslant C_7 \left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{t,B} \left(\int_B w^\beta\,dx \right)^{1/s\beta} \\
&\quad \leqslant C_8 \left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{t,B} |B|^{(1-\beta)/s\beta} \|w\|_{1,B}^{1/s} \\
&\quad \leqslant C_9 |B| \operatorname{diam}(B) \|u\|_{t,B} |B|^{(1-\beta)/s\beta} \|w\|_{1,B}^{1/s}.
\end{aligned} \tag{3.16}
$$

Now, select $m = s/r$, then $m < s$. By Lemma 2.3, we obtain

$$
\begin{aligned}
&\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_B \right\|_{s,B,w} \\
&\quad \leqslant C_9 |B| \operatorname{diam}(B) \|u\|_{t,B} |B|^{(1-\beta)/s\beta} \|w\|_{1,B}^{1/s} \\
&\quad \leqslant C_{10} |B| \operatorname{diam}(B) |B|^{(m-t)/mt} \|u\|_{m,\rho B} |B|^{(1-\beta)/s\beta} \|w\|_{1,B}^{1/s}.
\end{aligned} \tag{3.17}
$$

Using generalized Hölder inequality again, we find that

$$
\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} \left(|u| w^{1/s} w^{-1/s}\right)^m dx\right)^{1/m} \\
&\leqslant \left(\int_{\rho B} |u|^s w \, dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{m/(s-m)} dx\right)^{(s-m)/sm} \\
&= \|u\|_{s,\rho B,w} \|1/w\|_{1/(r-1),\rho B}^{1/s}.
\end{aligned}
\tag{3.18}
$$

Substituting (3.18) into (3.17), we have

$$
\begin{aligned}
\left\|T\left(G(u)\right) - \left(T\left(G(u)\right)\right)_B\right\|_{s,B,w} \leqslant{} & C_{10}|B| \operatorname{diam}(B) |B|^{(m-t)/mt} |B|^{(1-\beta)/s\beta} \\
& \times \|u\|_{s,\rho B,w} \|w\|_{1,B}^{1/s} \|1/w\|_{1/(r-1),\rho B}^{1/s}.
\end{aligned}
\tag{3.19}
$$

Notice that $w \in A_r(M)$, then

$$
\begin{aligned}
&\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{1/(r-1),\rho B}^{1/s} \\
&\quad \leqslant \left(\left(\int_{\rho B} w \, dx\right)\left(\int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/s} \\
&\quad = \left(|\rho B|^r \left(\frac{1}{|\rho B|}\int_{\rho B} w \, dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/s} \\
&\quad \leqslant C_{11}|B|^{r/s}.
\end{aligned}
\tag{3.20}
$$

Substituting (3.20) into (3.19), we obtain

$$
\begin{aligned}
&\left\|T\left(G(u)\right) - \left(T\left(G(u)\right)\right)_B\right\|_{s,B,w} \\
&\quad \leqslant C_{12}|B| \operatorname{diam}(B) |B|^{(m-t)/mt} |B|^{(1-\beta)/s\beta} |B|^{r/s} \|u\|_{s,\rho B,w} \\
&\quad \leqslant C_{13}|B| \operatorname{diam}(B) \|u\|_{s,\rho B,w}.
\end{aligned}
$$

We have proved that (3.7) is still true if $\alpha = 1$, this ends the proof of Theorem 3.3. □

4. The global $A_r(M)$-weighted inequalities

Finally, we are ready to prove the global Sobolev–Poincaré imbedding inequality and the global Poincaré inequality. We shall need the following lemma about the Whitney covers appearing in [10]. See [14] for more properties of Whitney cubes.

Lemma 4.1. *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$
\bigcup_i Q_i = \Omega, \quad \text{and} \quad \sum_{Q \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q}(x) \leqslant N \chi_\Omega(x)
$$

for all $x \in \mathbf{R}^n$ and some $N > 1$, where χ_E is the characteristic function for a set E.

Theorem 4.2. *Let M be a compact, oriented, C^∞ smooth, Riemannian manifold without boundary and $u \in L^p(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be an A-harmonic form on M. Assume that $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ is a homotopy operator defined in* (1.1) *and $w(x) \in A_r(M)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|T(G(u))\|_{p,M,w^\alpha} \leqslant C|M|\operatorname{diam}(M)\|u\|_{p,M,w^\alpha} \tag{4.1}$$

for any real number α with $0 < \alpha \leqslant 1$.

Proof. Since M is compact, there exists a finite coordinate chart cover $\{U_1, U_2, \ldots, U_m\}$ of M such that $\bigcup_{k=1}^m U_k = M$. We can equip M with a topology in unique way that each U_k is open, see [9, Chapter II]. Therefore, we may assume that U_k is open, $k = 1, \ldots, m$. Applying Lemma 4.1 to U_k(note that $\bigcup_{B\in\mathcal{V}} B = U_k$ now) and Theorem 2.8, we have

$$\begin{aligned}
\|T(G(u))\|_{p,U_k,w^\alpha} &\leqslant \sum_{B\in\mathcal{V}} \|T(G(u))\|_{p,B,w^\alpha} \\
&\leqslant \sum_{B\in\mathcal{V}} \big(C_1|B|\operatorname{diam}(B)\|u\|_{p,\rho B,w^\alpha}\big) \\
&\leqslant \sum_{B\in\mathcal{V}} \big(C_2|B|\operatorname{diam}(B)\|u\|_{p,\rho B,w^\alpha}\big)\chi_{\sqrt{\frac{5}{4}}\rho B}(x) \\
&\leqslant \big(C_2|U_k|\operatorname{diam}(U_k)\|u\|_{p,U_k,w^\alpha}\big) \cdot \sum_{B\in\mathcal{V}} \chi_{\sqrt{\frac{5}{4}}\rho B}(x) \\
&\leqslant C_k|M|\operatorname{diam}(M)\|u\|_{p,M,w^\alpha}.
\end{aligned} \tag{4.2}$$

From (4.2), we obtain

$$\begin{aligned}
\|T(G(u))\|_{p,M,w^\alpha} &\leqslant \sum_{k=1}^m \|T(G(u))\|_{p,U_k,w^\alpha} \\
&\leqslant \sum_{k=1}^m C_k|M|\operatorname{diam}(M)\|u\|_{p,M,w^\alpha} \\
&\leqslant |M|\operatorname{diam}(M)\|u\|_{p,M,w^\alpha} \cdot \sum_{k=1}^m C_k \\
&\leqslant C|M|\operatorname{diam}(M)\|u\|_{p,M,w^\alpha}.
\end{aligned} \tag{4.3}$$

This ends the proof of Theorem 4.2. □

Similar to the proof of Theorem 4.2, using Lemma 4.1 and Theorem 2.9, we have the following global weighted estimate for the composition of ∇, T and G.

Theorem 4.3. *Let M be a compact, oriented, C^∞ smooth, Riemannian manifold without boundary and $u \in L^p(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be an A-harmonic form on M. Assume that $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ is a homotopy operator defined in*

(1.1) and $w(x) \in A_r(M)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that

$$\left\|\nabla\left(T\left(G(u)\right)\right)\right\|_{p,M,w^\alpha} \leqslant C|M|\|u\|_{p,M,w^\alpha} \tag{4.4}$$

for any real number α with $0 < \alpha \leqslant 1$.

Theorem 4.4. *Let M be a compact, oriented, C^∞ smooth, Riemannian manifold without boundary and $u \in L^p(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be an A-harmonic form on M. Assume that $T : C^\infty(M, \wedge^l) \to C^\infty(M, \wedge^{l-1})$ is a homotopy operator defined in (1.1) and $w(x) \in A_r(M)$ for some $1 < r < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left\|T\left(G(u)\right)\right\|_{W^{1,p}(M),w^\alpha} \leqslant C|M|\|u\|_{p,M,w^\alpha} \tag{4.5}$$

for any real number α with $0 < \alpha \leqslant 1$.

Proof. Using (2.2), Theorems 4.2 and 4.3, we find that

$$\begin{aligned}\left\|T\left(G(u)\right)\right\|_{W^{1,p}(M),w^\alpha} &= \operatorname{diam}(M)^{-1}\left\|T\left(G(u)\right)\right\|_{p,M,w^\alpha} + \left\|\nabla\left(T\left(G(u)\right)\right)\right\|_{p,M,w^\alpha} \\ &\leqslant C_1 \operatorname{diam}(M)^{-1} \cdot |M| \operatorname{diam}(M)\|u\|_{p,M,w^\alpha} \\ &\quad + C_2|M|\|u\|_{p,M,w^\alpha} \\ &\leqslant C_1|M|\|u\|_{p,M,w^\alpha} + C_2|M|\|u\|_{p,M,w^\alpha} \\ &\leqslant C_3|M|\|u\|_{p,M,w^\alpha}.\end{aligned}$$

This ends the proof of Theorem 4.4. □

We should notice that the parameter α in theorems makes our results more flexible and powerful. If we select α to be some special values, we shall obtain the required inequalities. For example, choosing $\alpha = 1$ in Theorems 4.2, 4.3 and 4.4, respectively, we have the following corollary.

Corollary 4.5. *Let M be a compact, oriented, C^∞ smooth, Riemannian manifold without boundary and $u \in L^p(\Lambda^l M, w)$, $l = 1, 2, \ldots, n$, $1 < p < \infty$, be an A-harmonic form on a manifold M and $T : C^\infty(M, \Lambda^l) \to C^\infty(M, \Lambda^{l-1})$ be a homotopy operator defined by (1.1). Assume that $w \in A_r(M)$ for some $r > 1$. Then, there is a constant C, independent of u, such that*

$$\left\|T\left(G(u)\right)\right\|_{p,M,w} \leqslant C|M| \operatorname{diam}(M)\|u\|_{p,M,w}, \tag{4.6}$$

$$\left\|\nabla\left(T\left(G(u)\right)\right)\right\|_{p,M,w} \leqslant C|M|\|u\|_{p,M,w}, \tag{4.7}$$

$$\left\|T\left(G(u)\right)\right\|_{W^{1,p}(M),w} \leqslant C|M|\|u\|_{p,M,w}. \tag{4.8}$$

Remark. (1) From global results, we see that $T \circ G$ and $\nabla \circ T \circ G$ are bounded operators if M is bounded.

(2) These results can be used to study the integral properties of the compositions of the operators.

Similar to the proof of Theorem 4.2, using Lemma 4.1 and Theorem 3.3, we can prove the following global $A_r(M)$-weighted Poincaré inequality for the composition of T and G.

Theorem 4.6. *Let $u \in L^s(\Lambda^l M, w^\alpha)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be an A-harmonic form on a manifold M and $T : C^\infty(M, \Lambda^l) \to C^\infty(M, \Lambda^{l-1})$ be a homotopy operator defined by* (1.1). *Assume that $w \in A_r(M)$ for some $r > 1$. Then, there exists a constant C, independent of u, such that*

$$\left\| T\big(G(u)\big) - \big(T\big(G(u)\big)\big)_M \right\|_{s,M,w^\alpha} \leqslant C|M| \operatorname{diam}(M) \|u\|_{s,M,w^\alpha} \tag{4.9}$$

for any real number α with $0 < \alpha \leqslant 1$.

Remark. Similar to the global $A_r(M)$-weighted Sobolev–Poincaré imbedding inequality, for different choices of α in Theorem 4.6, we will have different versions of the global $A_r(M)$-weighted Poincaré inequality. Considering the length of the paper, we do not list these special global results here.

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